State Complexity of Population Protocols

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Introduction

The Setting

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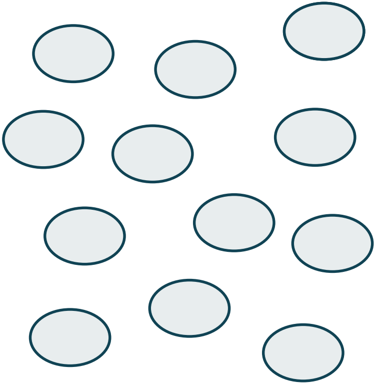

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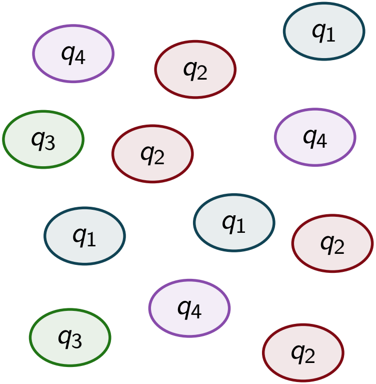

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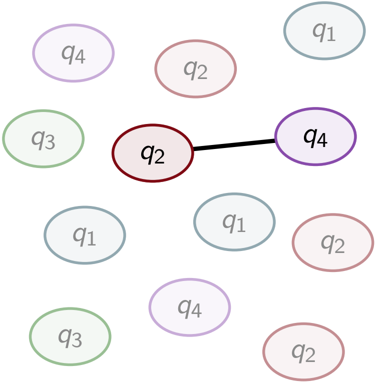

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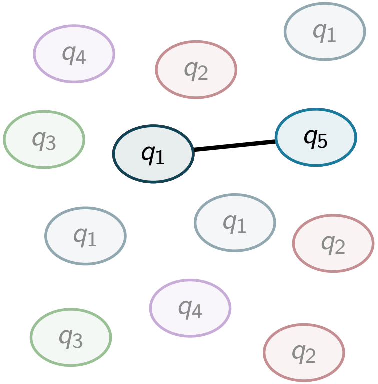

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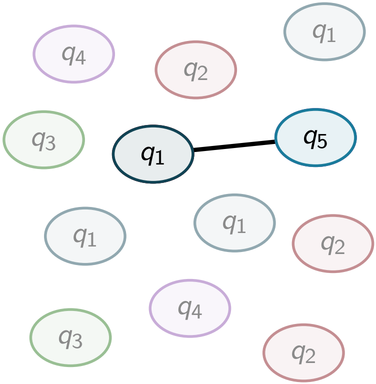

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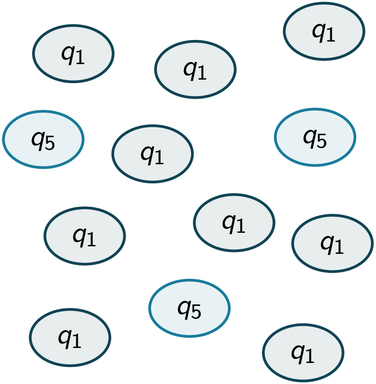
  - computation by *stable consensus*



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- Close to chemical reaction networks

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- decide exactly semi-linear (or Presburger) predicates
- can decide threshold: $\varphi(x) \Leftrightarrow x \geq k$, for any $k \in \mathbb{N}$

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- Time complexity in population protocols has a long history
  - many upper bounds, also lower bounds
- Little progress on space complexity until recently!

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- Count number of states depending on predicate.

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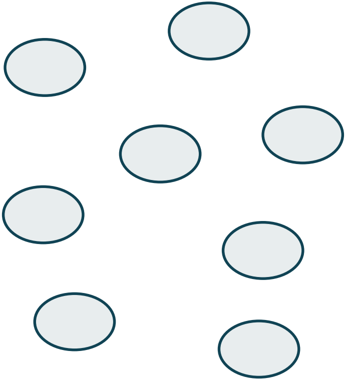
- Busy-Beaver question for population protocols!
- $\varphi(x) \Leftrightarrow x \geq 10^{23} \approx 2^{100}$

A simple protocol

- goal: decide $x \geq 3$
- idea: pebble counting



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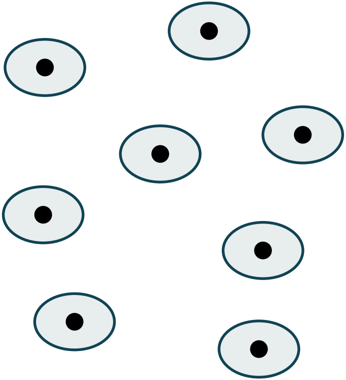

$I :=$ {  }

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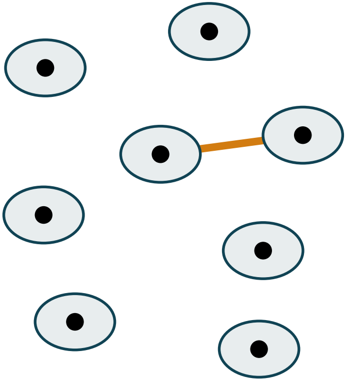

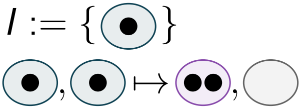

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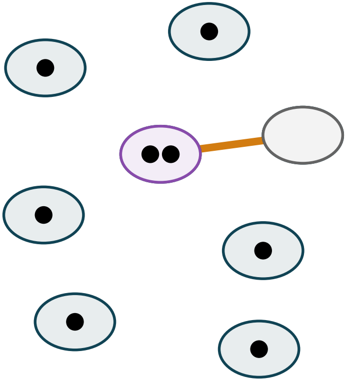

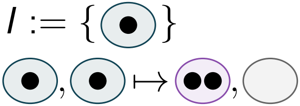

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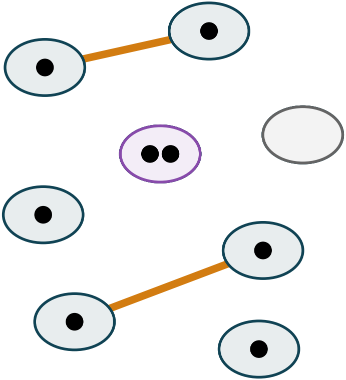

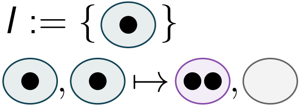

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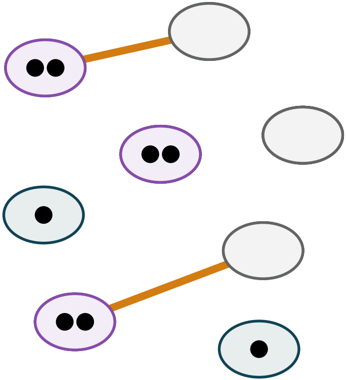

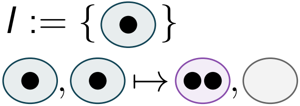

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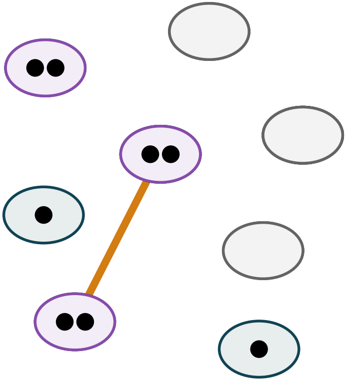

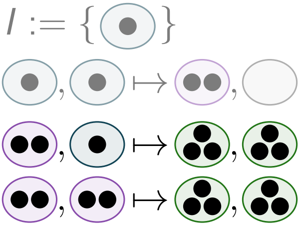

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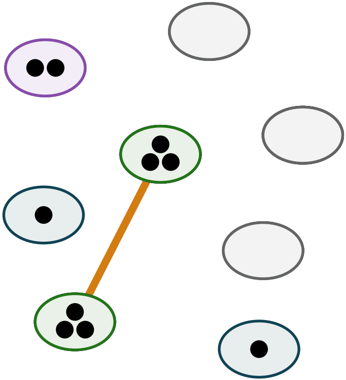

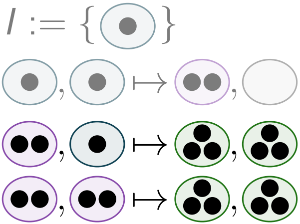

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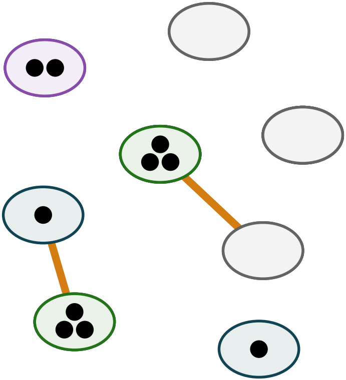

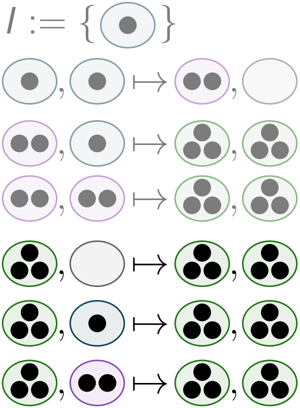

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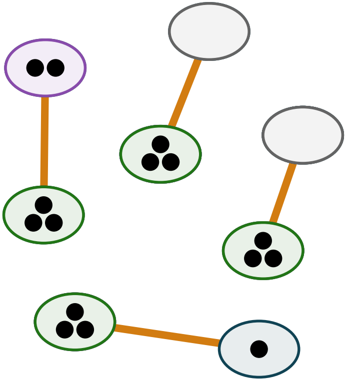

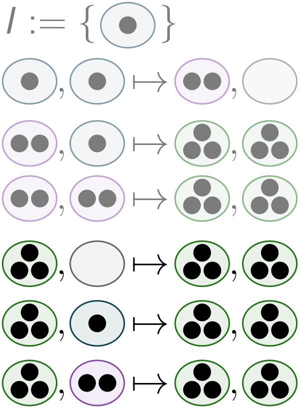

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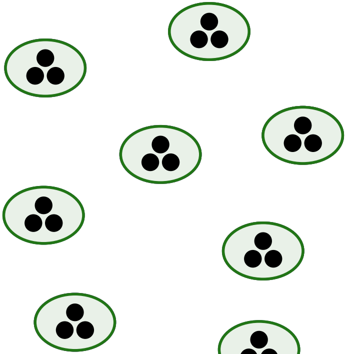

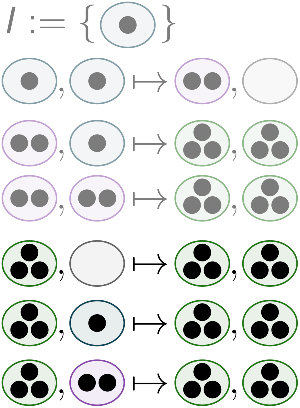

A simple protocol

- states $Q := \{0, ..., k\}$
- transitions

$$
\begin{aligned}
i, j &\mapsto i + j, 0 && \text{if } i + j < k \\
i, j &\mapsto k, k && \text{if } i + j \geq k
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- $\mathcal{O}(k)$ states

Can we do better?

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- $\mathcal{O}(\log \log k)$ states if leaders are provided [Blondin et al. 2018]
- $\mathcal{O}(\log \log k)$ states even without leaders! [Czerner 2022]

Can we do even better?

No!

No!

- $\Omega(\log \log \log k)$ states necessary without leaders [Czerner, Esparza 2021]

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Bounds are tight at $\Theta(\log\log k)$ states in both cases

What about other predicates?

General constructions

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- $\Omega(\text{poly}(\text{size}\,\varphi))$ states for any predicate φ [Blondin et al. 2020]
- optimally fast protocols with same bound [Czerner et al. 2022]

Thank you for
your attention!